GRIEVING WIDOWS

EXPLORING EXCESS MORTALITY FOLLOWING BEREAVEMENT

ABSTRACT

In this paper we show how there is a pronounced increase in an individual's chances of dying following the death of a partner. This is particularly the case within the first year following bereavement, and where bereavement occurs at the younger pensioner ages.

Allowing for the increased mortality experienced by widow(er)s reduces annuity prices by circa 50bp and should be of interest to insurers both in terms of the opportunity to gain a competitive edge in pricing, but also in the potential release of capital reserves.

The 'grieving widows effect', or the 'broken heart syndrome' as it is often referred to, is the phenomenon whereby the chances of dying are elevated immediate following the loss of a partner.

In many ways the existence of such an effect is no surprise. Companionship can convey a positive benefit on health and well-being through mutual care; and in later life a watchful eye to summon medical care quickly when needed. For example Friedman & Martin (2011) discusses the positive health benefits that companionship can bring.

Companionship also has a positive effect on an individual's desire to stave off a visit from the Grim Reaper. A striking example is a story we heard from Prof Tom Kirkwood. When interviewed, a couple participating in the University of Newcastle's 85+ study both independently responded that the driver keeping them alive is that they did not want to leave the other alone after their death.

In contrast the stress of losing a loved one removes this purpose for living. From a medical perspective this can manifest in a variety of ways including stress cardiomyopathy – a sudden weakening of the heart muscles induced by stress which can lead to acute heart failure – and through more gradual deterioration of function.

To date insurance companies and pension schemes assessing the financial costs of annuities have paid little attention to this effect. Yet in times of low interest rates annuity valuations are acutely sensitive to the longer dated cashflows, including those potentially payable via survivor benefits (i.e. the attaching annuity payable for life to a spouse who outlives the original annuity policyholder).

In this paper we show how allowing for this effect can help **reduce annuity prices by circa 50bp.** This benefit should be of interest to insurers both in terms of the opportunity to gain a competitive edge in pricing, but also in the potential release of capital reserves.

OUR ANALYSIS

Our analysis is based upon data collected by Club Vita in respect of annuitants within occupational pension schemes. Club Vita currently tracks around 1 in 10 of the UK retired population, reflecting the members of over 150 different occupational pension schemes. More information on Club Vita is included in Appendix A. The analysis in this paper relates to the data processed by Club Vita by 30 June 2012.

In analysing the 'grieving widows' effect we contrast mortality rates experienced amongst widows with women who are the 'first life' annuitant ('pensioner') and so more likely (but not guaranteed) to be married. To avoid distortions from retirees with mortality elevated by serious ill health we have restricted our attention to pensioners who retired in 'normal health' i.e. those who were not eligible for an enhanced benefit from their pension scheme owing to a known health condition at retirement.

QUANTIFYING THE GRIEVING WIDOWS EFFECT

The chart below compares the ratio of deaths observed in the Club Vita data for different age bands with the number of deaths which would have been expected had they been in line with a commonly used actuarial mortality table (known as an 'A/E ratio'). The table used is the PNFL00 tables as published by the CMI on behalf of the UK Actuarial Profession and cover pensioners who retired from occupational pension schemes where the benefits have been insured with a life office – specifically female pensioners who retired at or after their normal retirement age for that scheme.

The chart includes three lines representing widows, normal health (NH) pensioners and the combined data from both these groups. The dotted lines reflect the 95% confidence interval for the A/E ratio, calculated using the approximate formulae set out in Appendix B.

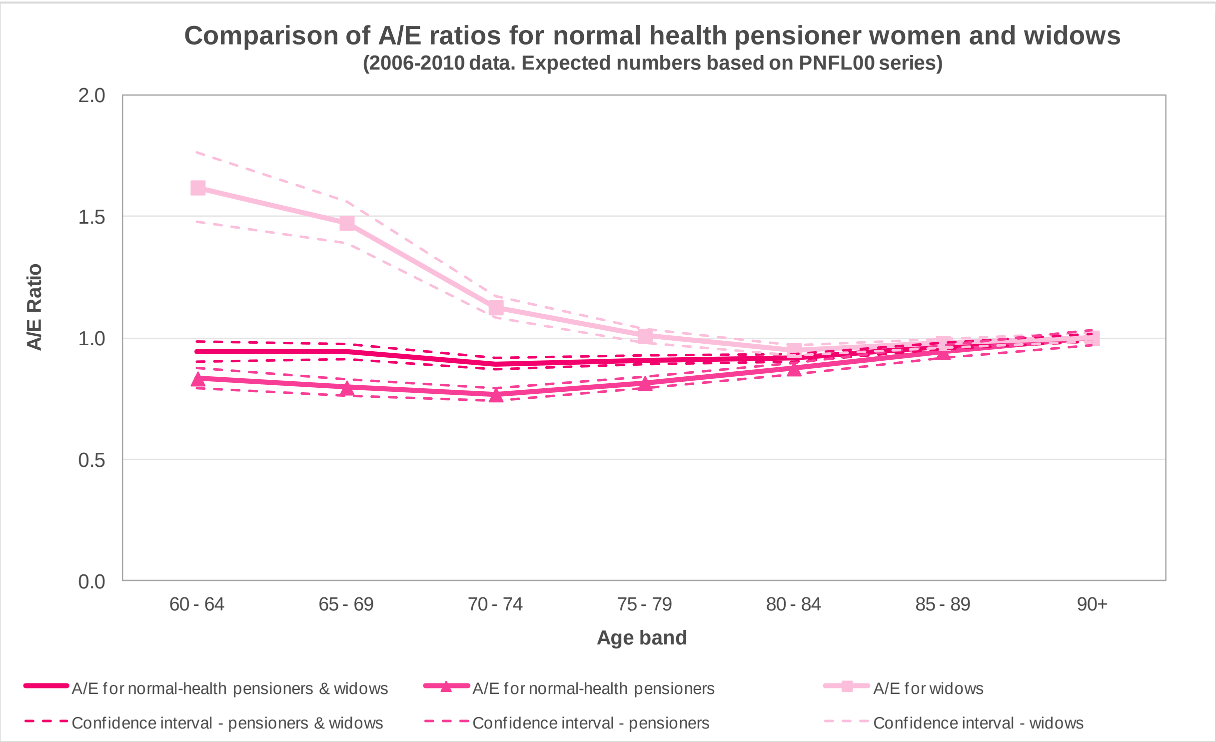
WE SEE THAT:

- There is a sharply higher death rate amongst young widows than female pensioners

  It is also worth noting though that the pensioner lines in the chart will include some widows i.e. where the woman is the beneficiary from the pension scheme and her husband has predeceased her. The proportion of pensioner beneficiaries who are widowed is likely to grow with age. As such, the true magnitude of the grieving effect is likely to be larger than that shown in the chart.

- There is convergence of relative mortality from age 70 upwards, suggesting that the mortality of widows is similar to that of female pensioners at older ages, but that the loss of a partner can lead to an increased chance of death at younger ages.

  This is in part due to the increasing proportion of pensioners who are themselves widows at older ages. However, the convergence of the relative mortality rates with age is also consistent with the compensation law of mortality whereby the relative differences in



mortality between different subpopulations within a species decrease with age as higher initial rates of mortality are compensated by slower increases with age (Gavrilov and Gavrilova, 1991). Furthermore, this is intuitive; mortality rates cannot exceed 1; so as they increase with age, so there is less scope for large ratios.

This suggests a strong grieving widows effect i.e. the mortality of widows being noticeably elevated as compared to female pensioners, and is consistent with other studies e.g. Spreeuw & Wang (2008). Hart et al (2007) suggests that this excess mortality manifests across most of the major causes including cardiovascular disease, coronary heart disease, stroke, cancer, and accidents (inc. violence); a conclusion which has been supported in a US study by Elwert & Christakis (2008).

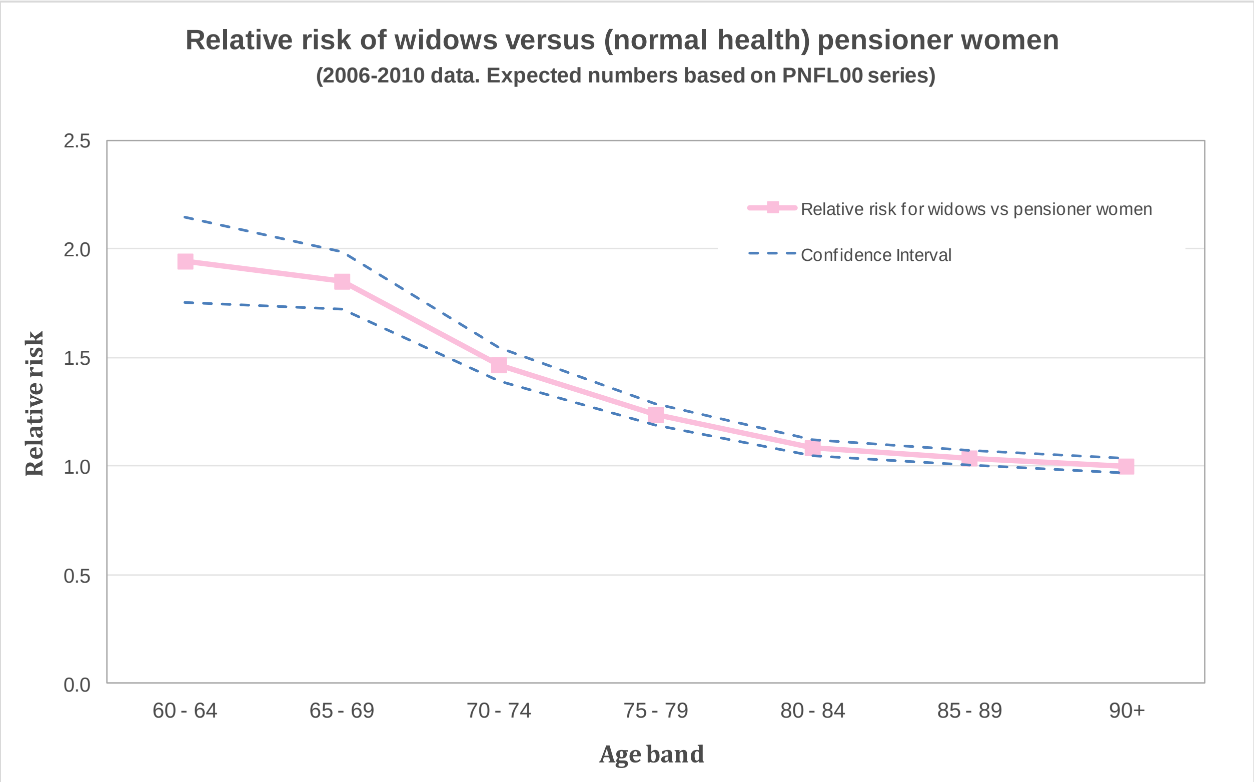
STATISTICAL CONFIDENCE

Since the confidence intervals shown above are non-overlapping for ages below 80, we can be very confident that the mortality rates are different at these younger ages. Indeed non-overlapping 95%confidence intervals gives us greater than 95% confidence in the null hypothesis that the mortality

rates for the different groups are different. However, at the older ages a more sophisticated test is needed to conclude whether the differences are significant. This can be tested by comparing *absolute* levels of A/E ratios (i.e. the difference in values) or the relative levels of the A/E ratios (i.e. the ratio).

In this paper we have elected to test the relative levels of the two groups, i.e. the ratio of their respective A/E ratios known as 'relative risk'. Whilst this leads to slightly more complex calculations of the confidence intervals it is more natural in the sense that it is more usual for adjustments to mortality rates to be expressed as percentage uplifts rather than flat additions.

The chart below plots the A/E ratio for the widows divided by the A/E ratio for the (normal health) female pensioners. To the extent that this relative risk is significantly different from 1 the two populations have differing mortality. To test this we have calculated an approximate 95% confidence interval for the relative risk (dotted lines) – the



derivation of which can be found in Appendix B.

We can see that below age 85 the confidence intervals do not include 1. Thus we can be confident that the mortality of widows below age 85 is statistically significantly different to that of female pensioners.

AGE, DURATION AND SELECTION

Whilst our analysis indicates a clear pattern with age, it is natural to ask whether this is the full story. For example:

1. The youngest widows will tend to have been recently widowed. At older ages the widows will be a mix of long-term widowed and the recently bereaved. As such the average duration since bereavement will increase with age and so the features seen above may be a **duration** rather than an age effect.
2. The younger widows will tend to have been married to a man who died at a relatively young age. This is likely to mean the first-life husbands were from the lower socio-economic groups who have higher mortality. In turn we would expect the widows to be from a similar background. As such the age pattern seen above could really be a **selection** effect whereby the younger widows have above average mortality owing to a bias towards less healthy socio-demographics.

We explore these possible issues further below.

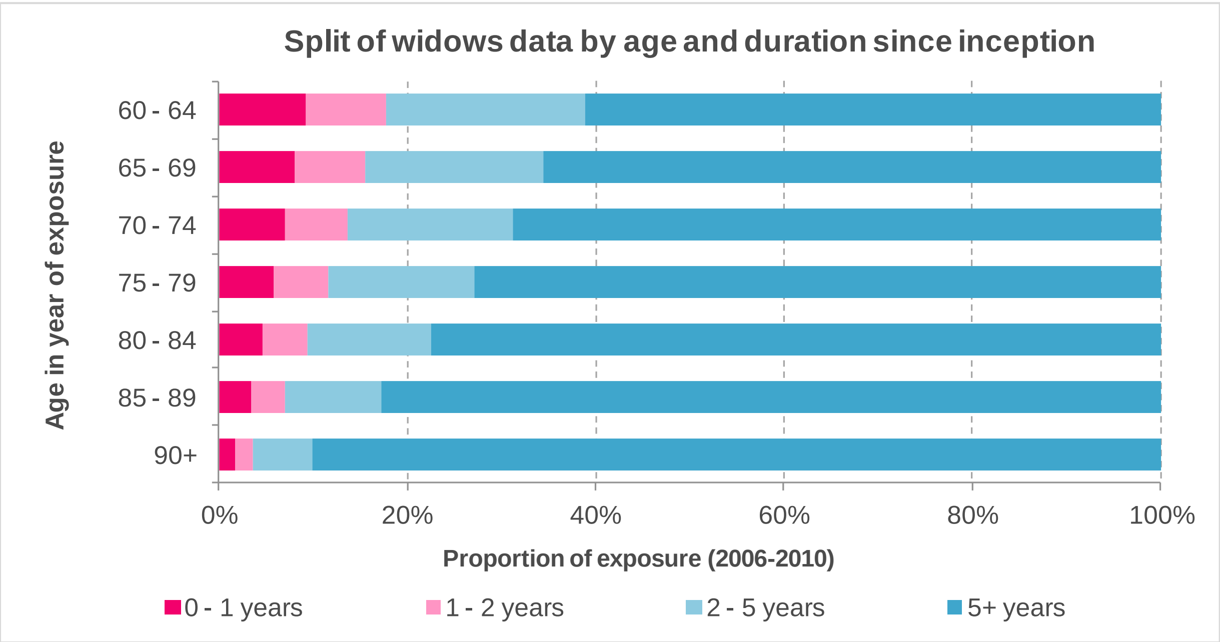
THE DURATION EFFECT

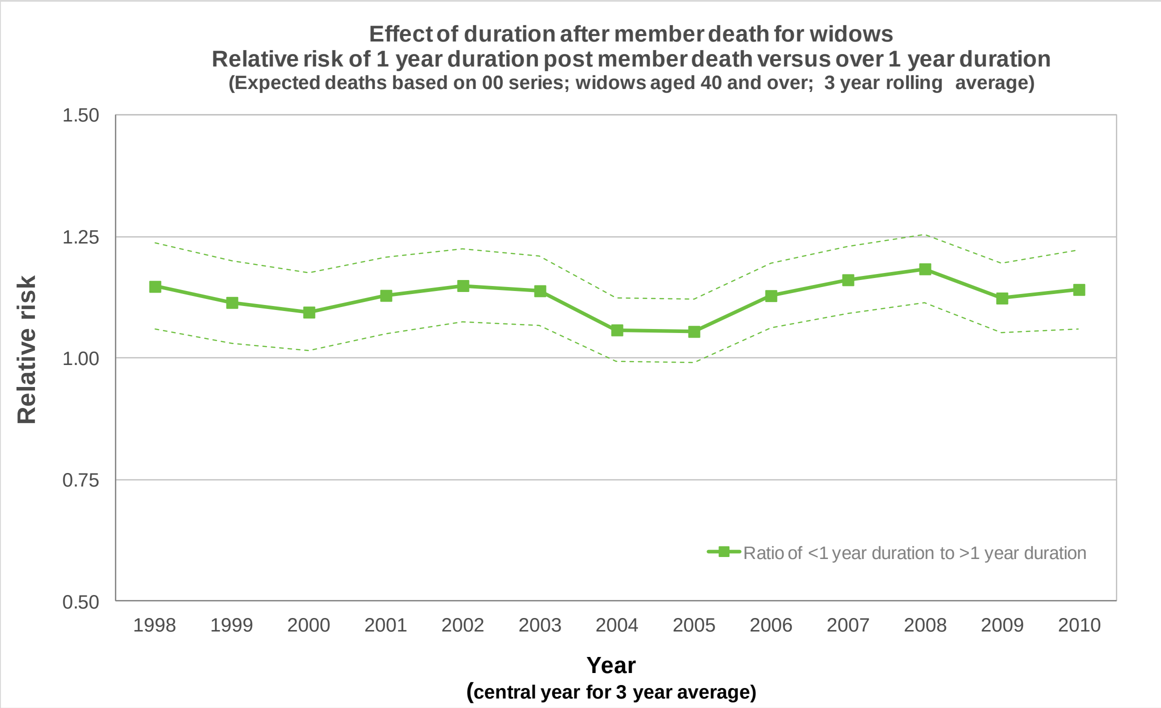
It is likely that younger widows have been most recently widowed, and as such most exposed to the negative effects of the grieving process. This is confirmed by the chart below which shows the split of exposures by the duration since the loss of their husband for each age group of the widow population.

WE SEE HOW:

- The proportion of widows pensions which have been in payment for less than one year falls from 9% for the youngest age group, to 2% for the over 90s
- The vast majority (90%) of widows pensions payable to widows aged over 90 have been in payment for more than 5 years (compared to 60% for widows aged 60-64)

In order to assess the impact that duration since bereavement has on mortality we have compared





the A/E ratio for those widows whose pensions have been in payment for less than one year, with those widows whose pensions have been in payment for more than one year. We have standardised these results with reference to the A/E seen for those widows where the duration is longer than 1 year to determine a *relative risk*.

For example a relative risk of 1.3 would suggest that mortality is typically 30% heavier amongst the recently bereaved widows than those who have been widowed for more than one year[1]. The chart above shows the result of this analysis for a series of three year exposure periods, centred on the year shown.

WE SEE THAT:

- The relative risk for widows whose pension has been in payment for less than one year is consistently greater than 1, indicating that there is an excess mortality at short durations
- The relative risk has fluctuated a little over time but generally been between 110% and 115% (i.e. mortality rates are elevated by around 10-15% in the year following bereavement)
- This relative risk is considerably lower than that seen between widows and pensioners at younger ages. As such we can reasonably infer that the grieving effect at younger ages is due to more than simply differences in duration mix i.e. there is ***both*** an age and a duration effect which influences the grieving 'penalty'

Using the methods described in Appendix B we have also been able to place an approximate 95% confidence interval around this relative risk; the dotted lines in this chart. Where this interval does

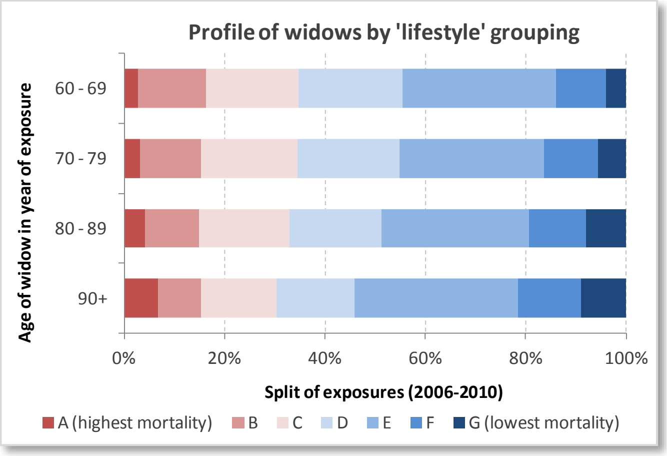
[1] Some readers may be wondering why we use relative risk rather than a simpler calculation such as the ratio of crude mortality rates. However crude mortality rates reflect the mortality at an 'average age' and so are only directly comparable when the underlying age distribution of the two populations are very similar. The earlier chart showing the distribution of duration of widowhood by age band shows that this is not the case here. The relative risk controls for this by standardising against 'expected mortality rates'.

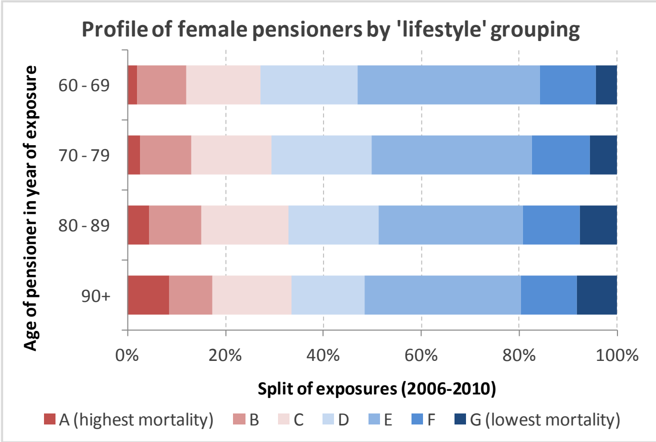
Whilst a relative risk of greater than 1 suggests higher mortality on average, this need not mean that the mortality is higher at all ages as the underlying mortality rates can have very different patterns with age. Separate analysis has confirmed that this is not an issue here.

not include 1 we can be confident that the short duration sub-population has materially different mortality to the longer duration widows' population i.e. the elevated mortality in the first year of payment is statistically significant. Since these confidence intervals exclude 1 for all calendar years except 2004 and 2005 we can have a high degree of confidence that mortality is elevated immediately following bereavement.

Further, by looking at the individual calendar years as independent observations of the relative risk we are able to determine a 95% confidence interval for the relative risk of (103%,122%) which gives us strong evidence that there is a clear elevation of mortality in the first year following the loss of a partner.

This finding of a clear duration effect to the post-bereavement mortality penalty is consistent with the findings of Jagger & Sutton (1991) which demonstrated substantially elevated mortality risk in the first 6 months following bereavement, before dropping away with duration. Their study was on a considerably smaller population (344 lives) and so the above provides much tighter bounds on the confidence interval. An older study in using close to 100,000 Swedish lives confirmed a substantial elevation in mortality for both widows and widowers in the three months following bereavement (Mellström et al, 1982); a result which was recently verified in a longitudinal study by Moon et al (2013).





THE SELECTION EFFECT

One way to consider if there is a selection effect whereby younger widows come from a materially different socio-demographic group to young female pensioners is to profile each group.

Within Club Vita we have developed seven socio-demographic longevity clusters, based on grouping postcodes with like lifestyles together into groups with distinct mortality and so longevity. In each case the lifestyle of an individual is assessed using their full UK postcode; linking this to third-party databases which profile the likely lifestyle of individuals living in that postcode using a combination of propensity scoring and consumer spending information. The resulting socio-demographic clusters are labelled A (highest mortality) through to G (lowest mortality). The charts above show the split by socio-demographic group of the widows and female ('normal health') pensioners underlying our earlier relative risk chart.

WE SEE THAT:

- A greater proportion of widows at the youngest ages lie in the bottom 3 lifestyle groups (A-C) compared to female pensioners(35% vs 27%) indicating there may be some slight selection effect taking place
- The proportion of individuals in the top lifestyle groups – particularly F and G – increases with age

  This is a natural consequence of the survivorship bias whereby the lower mortality of the higher socio-demographic groups results in a greater proportion of these groups surviving to an advanced age.

- The proportion of individuals in the bottom lifestyle group increases with age.

  This may seem counter-intuitive owing to the survivorship bias. However, this reflects the fact that care homes are most likely to be assigned to longevity group A, consistent with the increase in mortality rates expected upon loss of independence.

Whilst a small selection effect may be present it is unlikely to explain the age profile of the grieving effect seen earlier. Specifically, the gap between each of Club Vita's socio-demographic group above is around 8% in relative mortality terms. From this we can deduce that the differences in profile seen at ages 60-69 between widows and pensioners correspond to around 2% in relative mortality terms, compared to the 50% increase in mortality of widows compared to female pensioners.

This is further verified when we consider the relative risk of widows compared to female pensioners within each socio-demographic group in the chart below. For purposes of avoiding excess volatility we have grouped socio-demographic lifestyle groups A and B together, and F and G together.

WE CAN SEE THAT:

- Each socio-demographic group has a relative risk which follows a very similar profile with age to that seen earlier

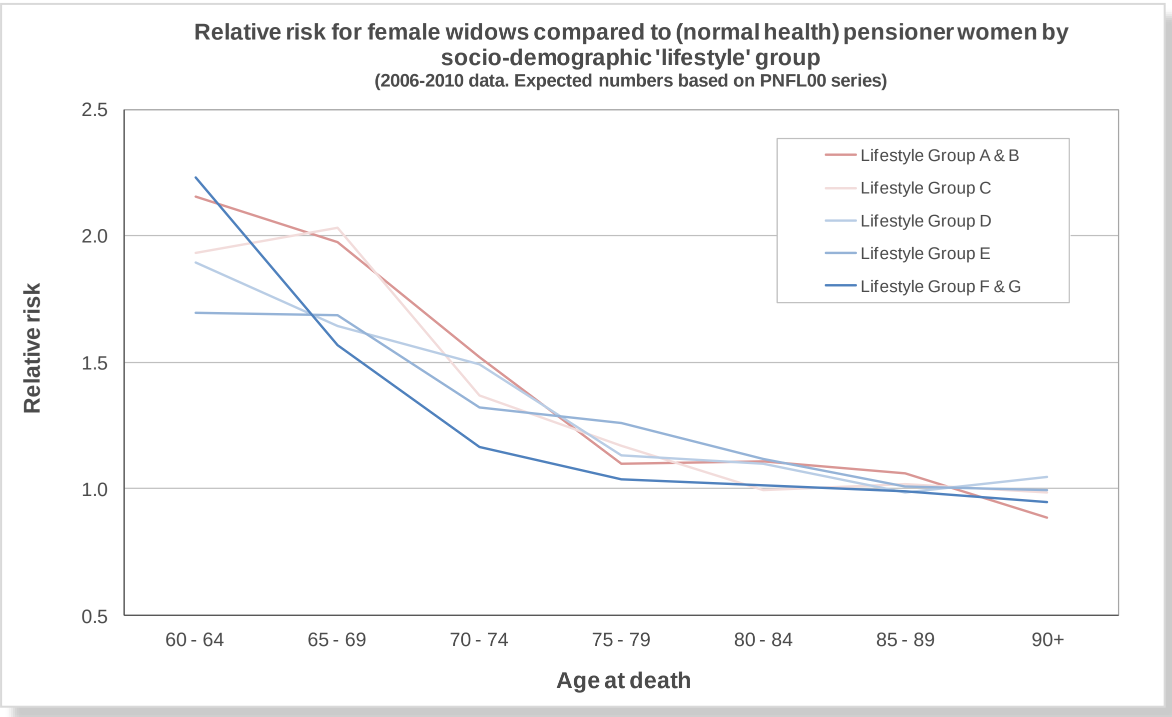
  This provides further comfort that what we are observing is a genuine age (and duration) effect, rather than an artefact of selection

- There is a weak pattern of the higher socio-demographic groups having less of a 'grieving penalty', suggesting that different socio-demographic groups may react differently to the grieving process

FINANCIAL MATERIALITY

Having illustrated the existence of the grieving widows effect a natural next step is to consider the potential financial significance of allowing for this in reserving and pricing of annuities.

To do this we have calculated the value of an annuity with an attaching spouse's pension under two different approaches. In each case the annuity is assumed to be payable at outset to a man. The approaches differ though in the mortality assumption used for the contingent spouse. Firstly, we have



assumed that the mortality of the spouse is in line with the 'average' female (normal health) pensioner within Club Vita's dataset. Secondly, we have assumed that the mortality of the spouse is in line with the 'average' female pensioner within Club Vita's dataset until she is bereaved, after which mortality is in line with the mortality of an 'average' widow within the Club Vita data.

Further, we have assumed that 90% of men are married at ages up to 65. For older annuitants we have assumed a lower proportion married are currently married consistent with an allowance for the wife having already pre-deceased the member. (No allowance has been made for the possibility of remarriage throughout our calculations.)

The chart below illustrates the results of our calculations for a range of different net discount rates, and using an attaching spouse's annuity of 2/3rds of the pension in payment. In each case the line represents the change in value of the annuity once the grieving effect is allowed for.

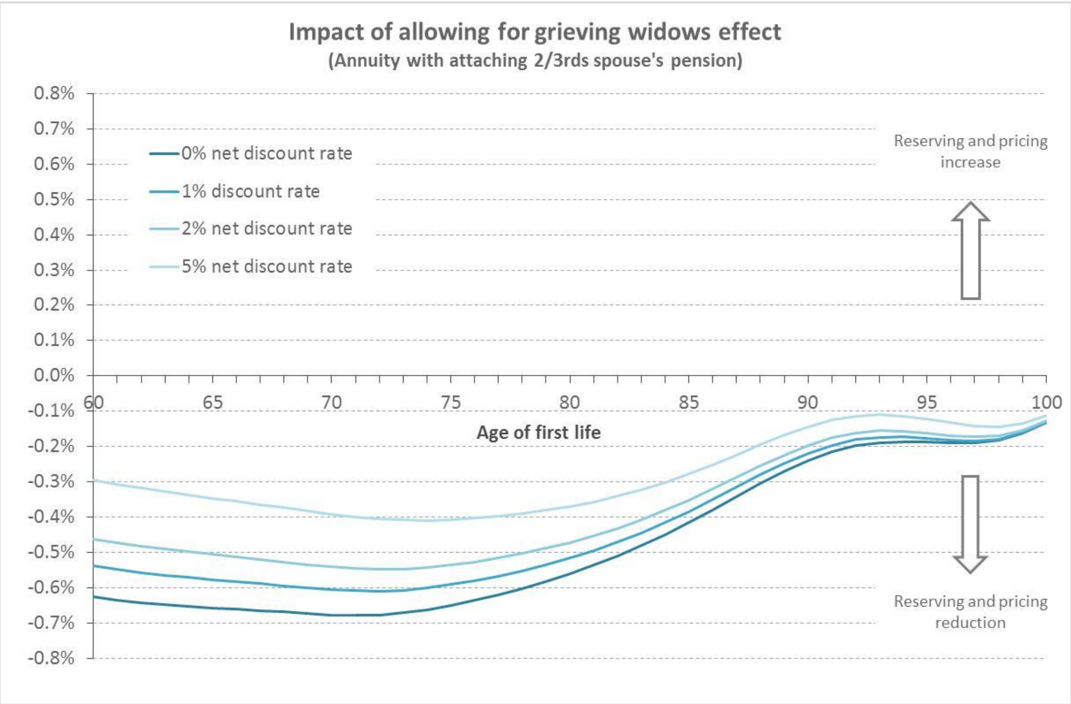
WE CAN SEE THAT:

- For all ages allowing for the grieving widows effect reduces the price of an annuity, by up to 70b.p. at inception where there is a low net discount and a substantial (2/3rds) attaching widows pension
- There is a clear shape with age – the grieving effect makes a material difference for annuitants aged under 75 (who will be the vast majority of the liabilities in a typical portfolio)
- Over the age of 75 the rapidly declining chance of still being married at the outset of the annuity results in a sharp decline in the price/reserving reduction from the grieving widows effect

Thus, **allowing for the grieving widows effect can be financially material**, especially for index-linked annuities in current financial times of close to zero, or indeed negative, net discount rates in the UK.

CONFOUNDING WITH SOCIO-ECONOMICS?

The astute reader might be concerned that by using female pensioner mortality for the spouse prior to the death of the original annuitant man we are confounding a grieving effect with a socio-economic effect. This would happen for example if the socio-economic status of the typical female pensioner is



different to that of the typical widow in the Club Vita data. The profiles of female pensioners and widows by lifestyle for different age bands shown on page 6 give us considerable comfort that this is unlikely to be the case. Further we have replicated our analysis of the financial impact using the mortality experienced for specific lifestyle groups. In each case we found that allowing for the grieving widows effect is financially material, albeit the precise quantum of the effect differs a little between the different lifestyle groupings.

WHAT ABOUT WIDOWERS?

In this paper we have focussed on widows. This is deliberate as Club Vita has far richer data on widows than widowers. This is a consequence of both the lower mortality of women leading to a greater proportion of women outliving their husbands and so greater data volumes for widows; and of UK pensions legislation where the requirement to provide a widowers pension was introduced after the requirement to provide a widows pension.

However, preliminary analysis of the Club Vita suggests similar conclusions to those presented above will apply to widowers. This would be consistent with Friedman & Martin (2011) who suggest that not only is a similar grieving effect seen amongst widowers, but that it is much more pronounced than for widows.

CONCLUSIONS

Thus we can conclude:

- There is a clear 'grieving widows' effect whereby mortality of widows is higher than that of female (normal health) pensioners
- This effect has both:
  - An age component, with mortality some 50% higher at younger ages, declining to close to parity at 90+
  - A duration effect where widows mortality rates are some 10-15% higher during the first year following bereavement compared to later durations
- The effect has the potential to reduce annuity prices. Using a second life mortality assumption for the widows benefit during payment which captures the grieving effect (rather than a generic female pensioner assumption) has the potential to reduce typical annuity prices by c. **50b.p.**
- For UK life companies with circa £200bn exposure to annuities (net of reinsurance) we estimate that allowing for this effect could release reserves of up to **£200m.**

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APPENDIX A: CLUB VITA

CLUB VITA LLP

Established in 2008, Club Vita LLP is the wholly owned sister company of Hymans Robertson LLP and leading provider of longevity analytics to UK occupational pension schemes.

THE DATASET

The Club Vita database is drawn annually from the administration systems of over 150 large occupational pension schemes. This dataset ('VitaBank') contains historic data on pensioner mortality from the early 1970s onwards, with substantial (and statistically meaningful) volumes from the early 1990s onwards. With over 150 participating pension schemes, Club Vita has amassed a dataset spanning in excess of 1.2 million pensioners, 600,000 historic annuitant deaths and a further 4 million deferred annuitants. In practice, therefore, we have a diverse sample of around 10% of the UK's retired population.

All data undergoes rigorous data cleansing prior to entry to the dataset. As a result of these checks we are, for example, able to:

- Identify the date from which a scheme has a full record of deaths

  This ensures we only use a scheme's data from the point where we have full reporting of data.

- Identify whether the rating factor data we have is reliable, for example, that pension amounts are credible (given salary)

  This enable us to exclude records where there are concerns on data quality – this is done in a way that avoids introducing biases, for example, via excluding more death records than living records.

- Maximise the quality of data via:
  - Verification and correction of postcodes
  - Existence verification exercises *(we routinely verify that there are no incurred, but not recorded, deaths amongst the 85+ population)*

THE 'VITACURVES' RATING FACTOR MODEL

Since VitaBank is drawn directly from administration systems, we have detailed information on each and every individual. This means we are able to readily differentiate mortality rates, and improvements therein, by factors such as occupation, affluence, and postcode-based lifestyle factors.

Using this data we have developed a proprietary rating factor model whereby individual pension scheme members have age specific mortality rates determined by his or her specific characteristics. These mortality rates have been graduated from first principles using generalised linear models (GLMs) rather than relying on ad hoc adjustments to published tables. This ensures that the pattern of convergence of mortality with advanced age is correctly captured. Currently the rating factor model captures a spread of over 10 years in life expectancy from age 65.

The rating factor model ('VitaCurves') is fully documented and the methods have been published and peer reviewed by the UK actuarial profession[2]. **VitaCurves are available to be used by insurance companies on a 'one-off' or licensed basis**; for further information contact Steven Baxter.

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[2] http://www.actuaries.org.uk/research-and-resources/documents/what-longevity-predictors-should-be-allowed-when-valuing-pension-sc

APPENDIX B: CALCULATION OF CONFIDENCE INTERVALS

AN INDIVIDUAL

In the simplest form for each individual aged x exact we are observing the Bernoulli random variable X that he or she dies within the next year.

$$X \sim Bernoulli(q_x) \qquad \text{(B1)}$$

A COLLECTION OF INDIVIDUALS

In practice we observe a collection of individuals.

- Let D_x be the random variable deaths amongst individuals aged x exact
- Then in the absence of censoring and truncation we have:

$$D_x \sim Bin(n_x, q_x) \qquad \text{(B2)}$$

Where:

- $n_x = \sum_{all\ individuals} I_{x_i = x}$
- x_i is the age of the i-th individual
- $I_{x=y} = 1$ if, and only if, $x = y$

Provided that we observe sufficiently many individuals (i.e. n_x is large) then we can reasonably approximate:

$$D_x \approx N(n_x q_x, n_x q_x (1 - q_x)) \qquad \text{(B3)}$$

This approximation also assumes that the individuals are independent, identically distributed random variables. In practice this assumption does not strictly hold for two reasons:

1. Deaths of individuals are not truly independent events – pandemics, natural disasters and harsh winters are just some of the reasons which can systematically influence outcomes across individuals.
2. Individuals have different underlying mortality rates depending on a wide range of factors such as affluence, lifestyle, genetics etc... As the q_x parameter differs from individual to individual the identically distributed assumption does not hold.

Accordingly the variance is likely to be a little greater than indicated by (B3) and so our confidence intervals will be slightly understated. The clear-cut nature of the conclusions drawn in the main body of this paper mean we have no concern that this approximation has led to false conclusions.

WORKING WITH CALENDAR YEARS

In practice we are looking over calendar years and so we are interested in the random variable $D_{x,t}$, the deaths during calendar year t amongst individuals aged x exact.

$$D_{x,t} \approx N(n_{x,t} q_{x,t}, n_{x,t} q_{x,t} (1 - q_{x,t})) \qquad \text{(B4)}$$

As mortality can also vary from calendar year to calendar year we need to also index our mortality rates by both age and time, hence the $q_{x,t}$ term.

Once we bring in individual years we need to accept that truncation happens, along with censoring. In order to broadly control for this we can replace $n_{x,t}$ in the above with $ETR_{x,t}$ the *initial exposed to risk* amongst individuals aged x (nearest birthday) at 1 January of the calendar year t (and use q_x values relating to age x exact at the start of the calendar year).

We then have that:

$$D_{x,t} \approx N\left(ETR_{x,t} q_{x,t}, ETR_{x,t} q_{x,t} (1 - q_{x,t})\right) \qquad \text{(B5)}$$

LOOKING OVER SEVERAL CALENDAR YEARS

In the calculations in the main body of this paper we are looking at the actual deaths over a series of consecutive calendar years from t_0 to t_1 i.e. our random variable deaths amongst those aged x can then be expressed as:

$$D_x(t_0,t_1) = \sum_{t=t_0}^{t_1} D_{x,t} \qquad \text{(B6)}$$

Provided the individual calendar year observations are independent then we have:

$$D_x(t_0,t_1) \approx N\big(E_x(t_0,t_1), V_x(t_0,t_1)\big) \qquad \text{(B7)}$$

Where:

$$E_x(t_0,t_1) \coloneqq \sum_{t=t_0}^{t_1} ETR_{x,t} q_{x,t} \qquad \text{(B8)}$$

$$V_x(t_0,t_1) \coloneqq \sum_{t=t_0}^{t_1} ETR_{x,t} q_{x,t}(1-q_{x,t}) \qquad \text{(B9)}$$

LOOKING OVER SEVERAL CALENDAR YEARS AND AGES

In our analysis we are concerned with the random variable of the number of deaths over a given time period, for a given range of ages, for example 65 to 69. In other words the random variable:

$$D(x_0,x_1,t_0,t_1) := \sum_{x=x_0}^{x_1} D_x(t_0,t_1) \qquad \text{(B10)}$$

If assume independence then:

$$D(x_0,x_1,t_0,t_1) \approx N(\mu\,,\sigma^2\,) \qquad \text{(B11)}$$

Where:

$$\mu := \sum_{x=x_0}^{x_1} E_x(t_0,t_1) \qquad \text{(B12)}$$

$$\sigma^2 \coloneqq \sum_{x=x_0}^{x_1} V_x(t_0,t_1) \qquad \text{(B13)}$$

The above assumes that the random variables $D_{x_0}(t_0,t_1),\ldots,D_{x_1}(t_0,t_1)$ are independent. In practice we know this does not hold if $x_1 > x_0$ and $t_1 > t_0$ as the individuals aged x_0 in t_0 who survive the year are then aged x_0+1 in year t_1. Consequently when $D_{x_0}(t_0,t_1)$ is large, we might expect $D_{x_0+1}(t_0,t_1)$ to be small owing to fewer survivors. This leads to an element of negative correlation between the random variables. Thus, the true variance is likely to be lower than indicated by (B11) i.e. works in the opposite direction to our normal approximation at (B2).

A CONFIDENCE INTERVAL FOR 'A/E'

The charts in the main body of this paper consider the confidence intervals for A/E ratios. The A/E ratio is defined as the observed value of $D(x_0,x_1,t_0,t_1)$, denoted $D_{obs}(x_0,x_1,t_0,t_1)$ divided by the expected observation for assumed values of $q_{x,t}$ i.e.:

$$\frac{A}{E} \coloneqq \frac{D_{obs}(x_0,x_1,t_0,t_1)}{\mathbb{E}(D(x_0,x_1,t_0,t_1) \setminus q_{x,t}=q^*_{x,t})} \qquad \text{(B14)}$$

Where $q^*_{x,t}$ are an assumed pattern of mortality rates with age. For the purposes of the analysis presented here $q^*_{x,t}$ are taken to be the mortality rates under the PNFL00 table published by the CMI without any overlay of improvements.

Defining $\mu^* = \mathbb{E}(D(x_0,x_1,t_0,t_1) \setminus q_{x,t} = q^*_{x,t})$ and $\kappa = \frac{\mu}{\mu^*}$ we then have:

$$\frac{A}{E} \approx N(\kappa, \frac{\sigma^2}{(\mu^*)^2}) \qquad \text{(B15)}$$

We can then estimate:

- κ by the unbiased estimator:

  $\hat{\kappa} = \frac{D_{obs}(x_0,x_1,t_0,t_1)}{\mu^*}$

- $\frac{\sigma}{\mu *}$ by the estimator $\hat{\sigma} = \frac{\sqrt{D_{obs}(x_0,x_1,t_0,t_1)}}{\mu^*}$

  Note that this a biased estimator which will overstate $\frac{\sigma^2}{(\mu^*)^2}$ *as it assumes that* $1 - q_x \approx 1$.

An approximate 95% confidence interval for the A/E ratio is then given by:

$$\hat{\kappa} \pm 1.96\,\hat{\sigma} \qquad \text{(B16)}$$

THE CONFIDENCE INTERVAL FOR RELATIVE RISK

The focus of this paper is on identifying whether different groups have statistically different levels of mortality relative to a reference table. If we consider two populations (for example female pensioners and widows) indexed 1 and 2 respectively then we are looking for differences between κ_1 and κ_2.

One way to consider this would be to look at:

$$\kappa_\Delta \coloneqq \kappa_1 - \kappa_2 \qquad \text{(B17)}$$

and test a null hypothesis that $\kappa_\Delta = 0$ against the alternate hypothesis that $\kappa_\Delta \neq 0$.

However, in practice within actuarial calculations we are usually more interested in ascertaining whether or not it would be appropriate to apply a scaling factor other than 100% (rather than applying a flat addition to mortality). Consequently we are more interested in the quantity:

$$\kappa_{\text{relative}} := \frac{\kappa_1}{\kappa_2} \qquad \text{(B18)}$$

and the associated null hypothesis that $\kappa_{\text{relative}} = 1$ versus the alternate hypothesis that $\kappa_{\text{relative}} \neq 1$.

Note that in the idealised situation where the two populations have the same age profile then $\kappa_{\text{relative}} = \frac{A_1}{A_2}$.

This leads us to a challenge though – how to determine the confidence interval for κ_{relative} based upon the observed value $\hat{\kappa}_{\text{relative}}$?

Thus, we are looking to develop a confidence interval for:

$$\kappa_{\text{relative}} := \frac{\kappa_1}{\kappa_2} = \mathbb{E}\left(\frac{A_1}{E_1}\right) \Big/ \mathbb{E}\left(\frac{A_2}{E_2}\right) \qquad \text{(B19)}$$

This is crucial as $\mathbb{E}\left(\frac{X_1}{X_2}\right)$ where $X_1 = \frac{A_1}{E_1}$ and $X_2 = \frac{A_2}{E_2}$ does not exist.

To see this note that X_1 and X_2 are normally distributed. Consequently X_2 has positive density in every neighbourhood of 0, and as

$$\int_{-\epsilon}^{\epsilon} \frac{1}{x} dx = \infty, \forall \epsilon > 0 \qquad \text{(B20)}$$

We can deduce that $\mathbb{E}\left(\frac{1}{X_2}\right)$ and $\mathbb{E}\left(\frac{X_1}{X_2}\right)$ do not exist.

Fieller's theorem provides us with a means of determining the confidence interval for κ_{relative} and takes into the account that values of $\mathbb{E}(X_2)$ below 1 have a bigger impact on κ_{relative} than values above 1 and thus we should have a non-symmetric confidence interval.

FIELLER'S THEOREM

Suppose that X and Y are bivariate normal random variables and we have n observational pairs $Z_i = (X_i, Y_i)$. Then the $(1-\alpha)$ confidence interval for $\mathbb{E}(X)/\mathbb{E}(Y)$ is given by:

$[-\infty, \infty]$	if	$q^2_{complete} < q^2$
$[-\infty, l_1] \cup [l_2, \infty]$	if	$q^2_{exc} < q^2 < q^2_{complete}$
$[l_1, l_2]$	otherwise	
		(B21)

Where:

$$q^2_{complete} := \frac{\hat{\mu}_1^2 \hat{c}_{22} - 2\hat{\mu}_1\hat{\mu}_2\hat{c}_{12} + \hat{\mu}_2^2\hat{c}_{11}}{\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2} \qquad \text{(B22)}$$

$$q^2_{exc} := \frac{\hat{\mu}_2^2}{\hat{c}_{22}} \qquad \text{(B23)}$$

$$q = t_{n-1,\alpha/2} \qquad \text{(B24)}$$

$$l_{1,2} = \frac{(\hat{\mu}_1\hat{\mu}_2 - q^2\hat{c}_{12}) \mp \sqrt{f}}{\hat{\mu}_2^2 - q^2\hat{c}_{22}} \qquad \text{(B25)}$$

$$f = (\hat{\mu}_1\hat{\mu}_2 - q^2\hat{c}_{12})^2 - (\hat{\mu}_1^2 - q^2\hat{c}_{11})(\hat{\mu}_2^2 - q^2\hat{c}_{22}) \qquad \text{(B26)}$$

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{(B27)}$$

$$\hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad \text{(B28)}$$

$$\hat{c}_{11} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_1)^2 \qquad \text{(B29)}$$

$$\hat{c}_{12} = \hat{c}_{21} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_1)(Y_i - \hat{\mu}_2) \qquad \text{(B30)}$$

$$\hat{c}_{22} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \hat{\mu}_2)^2 \qquad \text{(B31)}$$

Note that in the above $l_{1,2}$ us shorthand for l_1 and l_2 since they only differ in the choice of the $\mp$ respectively.

We do not attempt to provide a proof of Fieller's theorem here – however those wishing to understand this theorem in more depth are guided to Fieller's original paper (Fieller, 1954), Franz (2007) and especially von Luxburg & Franz (2004) which provides a geometric proof which is more readily visualised.

APPLICATION OF FIELLER'S THEOREM HERE

In the main body of this report we have deliberately chosen disjoint populations for our A/E variables. This has the direct consequence that

$$\hat{c}_{12} = \hat{c}_{21} = 0 \qquad \text{(B32)}$$

Further:

$$\hat{\mu}_1 = \hat{\kappa}_1,\, \hat{\mu}_2 = \hat{\kappa}_2,\, \hat{c}_{11} = \hat{\sigma}_1^2,\, \hat{c}_{22} = \hat{\sigma}_2^2 \qquad \text{(B33)}$$

in the notation of earlier.

This reduces the terms in (B22) to (B31) to:

$$q^2_{complete} := \frac{\hat{\kappa}_1^2\hat{\sigma}_2^2+\hat{\kappa}_2^2\hat{\sigma}_1^2}{\hat{\sigma}_1^2\hat{\sigma}_2^2} \qquad \text{(B34)}$$

$$q^2_{exc} := \frac{\hat{\kappa}_2}{\hat{\sigma}_2^2} \qquad \text{(B35)}$$

$$l_{1,2} = \frac{\hat{\mu}_1\hat{\mu}_2 \mp q\sqrt{\hat{\mu}_1^2\hat{\sigma}_2^2+\hat{\mu}_2^2\hat{\sigma}_1^2-q^2\hat{\sigma}_1^2\hat{\sigma}_2^2}}{\hat{\mu}_2^2-q^2\hat{\sigma}_2^2} \qquad \text{(B36)}$$

Defining:

$$\hat{\kappa}_{\text{relative}} := \frac{\hat{\kappa}_1}{\hat{\kappa}_2} \qquad \text{(B37)}$$

We then can rewrite (B36) as:

$$\frac{\hat{\kappa}_{\text{relative}}}{1-g} \mp t_{n-1,\alpha/2}.SE_{\hat{\kappa}_{\text{relative}}} \qquad \text{(B38)}$$

Where:

$$g = \left(t_{n-1,\alpha/2}\frac{\hat{\sigma}_2}{\hat{\mu}_2}\right)^2 \qquad \text{(B39)}$$

$$SE_{\hat{\kappa}_{\text{relative}}} = \frac{\hat{\kappa}_{\text{relative}}}{1-g}\sqrt{(1-g)\frac{\hat{\sigma}_1^2}{\hat{\mu}_1^2}+\frac{\hat{\sigma}_2^2}{\hat{\mu}_2^2}} \qquad \text{(B40)}$$

Under this formulation we can clearly see how the confidence interval is non-symmetrical about $\hat{\kappa}_{\text{relative}}$, as we would intuitively expect. Instead the confidence interval is symmetric about $\frac{\hat{\kappa}_{\text{relative}}}{1-g}$.

We can apply this formulation directly when we have more than 1 observation (i.e. $n > 1$) – for example we do this when considering 13 observations from individual calendar years when considering the duration effect.

WHEN N=1

In a number of instances we have shown a confidence interval where we have just one observable point, the relative actual vs expected statistics for a specific year or a specific age band.

For these years we have unapologetically abused statistics a little to be able to produce our confidence interval. Here we have noted that were we to be able to take a large number of repeated observations then we would have an asymptotically normal confidence interval, since as $n \to \infty$ so $t_n \to N(0,1)$. Thus an approximate confidence interval for the relative risk is:

$$\frac{\hat{\kappa}_{\text{relative}}}{1-g} \mp z_{\alpha/2}.SE_{\hat{\kappa}_{\text{relative}}} \qquad \text{(B41)}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile point of the standard normal distribution.

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